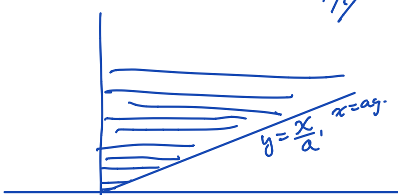
# Lecture 25

**ex:** Suppose that the joint density of $X$ and $Y$ is given by
$$f(x,y) = \begin{cases} e^{-(x+y)} & 0<x<\infty,\ 0<y<\infty \\ 0 & \text{Otherwise.} \end{cases}$$

Find the density function of $X/Y$.

**Soln:** First, we find the cumulative distribution function of $X/Y$: For $a>0$, we have
$$F_{X/Y}(a) = P\left\{\frac{X}{Y} \le a\right\}$$




$$\{(x,y): \tfrac{x}{y} \le a\}$$
$$\shortparallel$$
$$\{(x,y): y \geq \tfrac{x}{a}\}$$

$$= \iint_{x/y \le a} e^{-(x+y)}\,dx\,dy$$
$$= \int_0^\infty \int_0^{ay} e^{-(x+y)}\,dx\,dy.$$
$$= \int_0^\infty e^{-y} \int_0^{ay} e^{-x}\,dx\,dy$$
$$= \int_0^\infty e^{-y}(1-e^{-ay})\,dy.$$
$$= -e^{-y} + \frac{e^{-y(a+1)}}{a+1} \Bigg|_0^\infty$$
$$= 1 - \frac{1}{a+1}$$

Differentiating, we get $f_{X/Y}(a) = \dfrac{1}{(a+1)^2}$

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We can also define the joint probability of $n$ random variables:

" $F(a_1,\dots,a_n) := P(X_1 \le a_1, \dots, X_n \le a_n)$.

$X_1,\dots,X_n$ are said to be jointly continuous if there is a function $f(x_1,\dots,x_n)$ such that for some set $C \subseteq \mathbb{R}^n$,

$$P((X_1,\dots,X_n)\in C) = \iint_C \cdots \int f(x_1,\dots,x_n)\,dx_1,\dots,dx_n.$$

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Ex: Suppose we have an experiment with $r$ possible outcomes, with the $i$th outcome occuring with probability $p_i$, each mutually exclusive, $\sum_{i=1}^{r} p_i = 1$. Suppose that we perform the experiment $n$ times. Let $X_i$ be the number of experiments with outcome $i$. Then the joint distribution

$$P(X_1=n_1,\dots,X_r=n_r) = \frac{n!}{n_1!\cdots n_r!} p_1^{n_1} p_2^{n_2}\cdots p_r^{n_r}$$

when $\sum n_i = n$, is called the multinomial distribution.

when $r=2$, then we have the binomial distribution.

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# Independent Random variables.

Defn: We say that RVs $X$ and $Y$ are independent iff for any sets $A, B \subseteq \mathbb{R}$,

$$\underbrace{P(X \in A, Y \in B)}_{= P((X,Y) \in A \times B)} = P(X \in A) P(Y \in B)$$

i.e. the events "$X \in A$" and "$Y \in B$" are independent for any $A, B \subseteq \mathbb{R}$.

It can be shown that $X$ and $Y$ are independent iff, for all $a, b \in \mathbb{R}$,

$$P(X \leq a, Y \leq b) = P(X \leq a) P(Y \leq b).$$

Thus, if $F(x,y)$ is the joint cdf of $X$ and $Y$, then $X$ and $Y$ are independent iff

$$F(x,y) = F_X(x) F_Y(y),$$

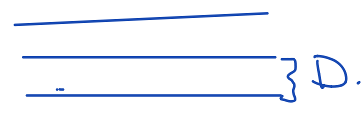
where $F_X(x)$ and $F_Y(y)$ are the marginal cdf's of $X$ and $Y$, respectively ($\frac{d}{dx} F_X(x) = f_X$, $\frac{d}{dy} F_Y(y) = f_Y$ if we are in the continuous setting).

If $X$ and $Y$ are discrete, then $X$ and $Y$ are independent <u>iff</u> $p(x,y) = p_X(x) p_Y(y)$ (see text for verification).

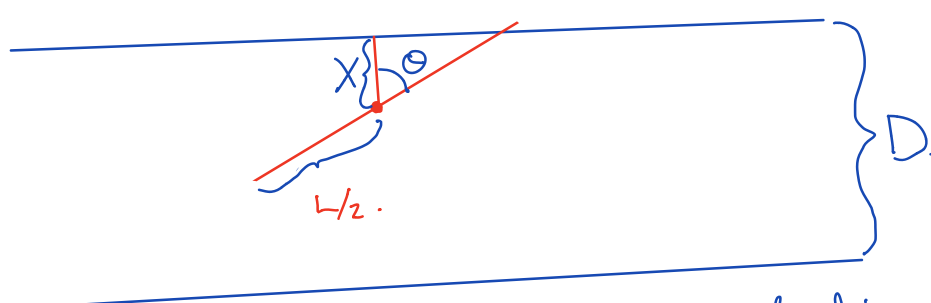
If $X$ and $Y$ are jointly continuous, we have an analogous result: $X$ and $Y$ are independent iff $f(x,y) = f_X(x)\, f_Y(y)$.

**Example** (Buffon's Needle).

- Suppose we have a table ruled with equidistant lines, each distance $D$



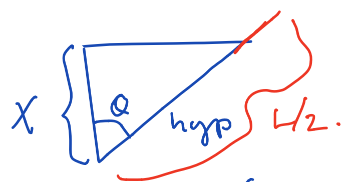

- A needle with length $L \leq D$ is thrown on the table.
- What is the probability that the needle intersects a line (if it does not intersect a line, it must be contained in a strip between 2 lines.
- we specify the position of the needle as follows:




↳ $X$ is the distance from the mid point of the needle and the closest line.

↳ $\theta$ is the angle between the segment joining the middle of the needle to the closest line (of length $X$) and the needle itself.

Consider the triangle cut out by the needle, the perpendicular and the line:




- Then $\cos(\theta) = \dfrac{X}{hyp}$, so $hyp = \dfrac{X}{\cos\theta}$

- Observe that the needle intersects the line iff $hyp = \dfrac{X}{\cos\theta} < \dfrac{L}{2}$. i.e. $X < \frac{L}{2}\cos\theta$.

- Now, $0 \le X \le D/2$ and $0 \le \theta \le \pi/2$, and $X$ and $\theta$ are uniform random variables.

Therefore: $P(X < \frac{L}{2}\cos\theta) = \displaystyle\iint_{x < \frac{L}{2}\cos(y)} f_X(x) f_\theta(y)\,dx\,dy$

$$= \iint_{x < \frac{L}{2}\cos(y)} \left(\frac{2}{D}\right)\left(\frac{2}{\pi}\right) dx\,dy$$

$$= \frac{4}{D\pi} \iint_{x < \frac{L}{2}\cos(y)} dx\,dy = \frac{4}{D\pi}\int_0^{\pi/2}\int_0^{L/2\cos(y)} dx\,dy = \frac{4}{D\pi}\int_0^{\pi/2} \frac{L}{2}\cos y\,dy$$

$$= \frac{2L}{\pi D}.$$

Let $p = P(\text{needle intersects a line})$. Then

$$p = \frac{2L}{\pi D} \Rightarrow \pi = \frac{2L}{pD}.$$

This gives us a way of estimating $\pi$! Throw a need on the table 100 times. If $N$ is the # of times the needle crosses the line, then $p \approx \frac{N}{100}$,

so $\pi \approx \frac{2L}{D}\left(\frac{100}{N}\right)$.